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# One-dimensional Casimir effect perturbed by an external field

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**Abstract.** We consider the problem of evaluating the Casimir effect by the mode-sum method for a quantum field in a one-dimensional space, in the presence of two point-like boundaries of Dirichlet type, and under the influence of a constant external field—which may be envisaged as a classical gravitatory field near the surface of a planet or a classical electric field in the interior of a flat capacitor. The case of infinitely separated points is also examined. Despite apparent simplicity, the calculation exhibits rather non-trivial aspects. The possibility of an experimental observation of these effects is considered.

## 1. Introduction

Some features of quantized fields can be studied in terms of the response of a vacuum to external fields—electric or gravitatory. In [1] special attention was paid to the ground state and vacuum polarization, in the presence of an external electric field, but relatively little was said about the actual values of the vacuum energy which—as commented, for example, in [2] by the same authors—can in principle be found using the mode-sum method. This, as well as other evaluation techniques, are described in [3]. In the present work we point at the possibility of numerically evaluating a quantity of this type by zeta-function regularization. With the final aim of addressing the problem of the gravitational influence on the original Casimir effect, we give here a first example based on a toy model, namely a neutral scalar field in a $(1+1)$-dimensional spacetime, subject to Dirichlet boundary conditions and under the effect of a static external field linearly entering the evolution equation. As observed in [1], an example of this nature may indeed be useful for later investigation with a realistic model. Being more precise, our system will differ from the one in [1] in that the field discussed there was a charged one, while ours is not, and also in the linear (not quadratic) way in which the potential is added to the field equation.

Relationships between gravity and the Casimir effect have been highlighted in other contexts; for example, the authors of [4] found the contribution to the effective potential generated by a single graviton loop on a background manifold of Kaluza–Klein type $M^4 \times S^N$. A crucial element in these studies was the theory of harmonics on the $N$-sphere, which yields Laplacian eigenvalues with relatively easy expressions, i.e. polynomials in the relevant quantum numbers. Despite the simplicity of our own settings—Euclidean $(1+1)$-dimensional spacetime and interaction just from an external source—this type of dependence

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no longer takes place. The eigenfrequencies happen to be solutions of transcendental equations, turning the evaluation of the spectral zeta function into a serious issue.

After assigning reasonable values to the physical magnitudes of the problem, the final numbers obtained indicate the interest of continuing the discussion with a more realistic (although more difficult) situation. Hopefully, this might lead to the devising of some experiment for distinguishing the cases of zero and non-zero external field. Also in connection with this type of system, the problem of instabilities (and related issues such as the Klein paradox [5]) coming from complex eigenvalues would be expected to appear in the strong external field regime. Since in our system we keep this field relatively weak, instabilities should not play any significant role (see, e.g., comments in [1]).

The calculation method to follow will be a variant of the one used in [6, 7], where a given analytic continuation technique plus some numerical effort made the successful application of the complete spectral zeta function possible for evaluating vacuum energies in problems with spherical symmetry and non-polynomic spectrum. This approach rests on ideas along the same lines of reasoning as in [8]. Particularly, the second of these studies contains a proposal about the use of this analytic-continuation philosophy in the study of the Hartle–Hawking wavefunction of the universe, thus bringing this type of thinking into the realm of modern quantum cosmology. The development of one-loop approaches from the analysis of boundary problems in this context (see, e.g., [9]) has proven to be a fruitful ground for the application of zeta-function methods [10]. Similar ideas have been further exploited with various aims in [11], on the underlying basis of the Seeley–De Witt series for Laplacian operators.

## 2. Chargeless scalar field

A quantum scalar field under the influence of an external effective potential can be described by means of an equation of the type

$$(\Box - m^2 + V(x))\Phi(t,x) = 0 \tag{2.1}$$

where $\Box$ is the one-dimensional d'Alembertian, $\Box = (\partial^2/\partial t^2) - (\partial^2/\partial x^2)$. After separating the time dependence of the solution, as $\Phi(t,x) = \mathrm{e}^{-\mathrm{i}\omega t}\phi(x)$, the equation for $\phi(x)$ reads

$$\phi'' + (\omega^2 + m^2 - V(x))\phi = 0. \tag{2.2}$$

In view of the way it enters our field equation, this $V(x)$ may be interpreted—perhaps more correctly—as the description of an object responsible for *boundary semihardening* [12], rather than a potential in the ordinary sense (dimensionally speaking, it is the square of an energy). Particularly, we will consider the linear form $V(x) = Ex$, having in mind, for example, $E = q\epsilon$ electric or $E \propto mg$, of gravitational style. The simplifying assumption $m^2 = 0$ yields

$$\phi'' + (\omega^2 - Ex)\phi = 0. \tag{2.3}$$

After making the variable change (for each value of $\omega$)

$$y_\omega = \frac{Ex - \omega^2}{E^{2/3}} = E^{1/3}x - \lambda_\omega \qquad \lambda_\omega \equiv \frac{\omega^2}{E^{2/3}} \tag{2.4}$$

the transformed equation is (calling $y_\omega = y$ again)

$$\frac{\mathrm{d}^2}{\mathrm{d}^2 y}\phi - y\phi = 0. \tag{2.5}$$

Its general solution, written in terms of Bessel functions, has the form

$$\phi(y) = y^{1/2}[c_1 J_{1/3}(\tfrac{2}{3}(-y)^{3/2}) + c_2 J_{-1/3}(\tfrac{2}{3}(-y)^{3/2})] \tag{2.6}$$

where $c_1$ and $c_2$ are to be determined by the boundary conditions. An alternative possibility is to note that, since (2.5) is the Airy equation, we may also express its general solution in terms of Airy functions $\phi(y) = \bar{c}_1 \mathrm{Ai}(y) + \bar{c}_2 \mathrm{Bi}(y)$. The mathematical expressions simplify greatly, but the degree of numerical difficulty is quite similar and the ensuing discussion changes little. Furthermore, in the case that the potential enters the equation quadratically—which is reviewed in the next section—the general solution involves Bessel functions of an index different from 1/3 and the reduction to Airy functions no longer applies. We should also bear in mind that each $y = y_\omega$ depends on $\omega$.

### *2.1. Case of Dirichlet boundary conditions at $x = 0$ and $x = L$*

In this first situation, we shall require vanishing conditions at the endpoints of a finite space interval. In the language of [12], this amounts to adding the imposition of *hard boundary* conditions. By the change (2.4), $x = 0 \Rightarrow y_\omega = -\lambda_\omega$, and $x = L \Rightarrow y_\omega = -(\lambda_\omega - \overline{L})$, where $\overline{L} \equiv E^{1/3}L$ is dimensionless. Thus, the system to be solved is

$$\phi(y = -\lambda_\omega) = 0 \qquad \phi(y = -(\lambda_\omega - \overline{L})) = 0 \tag{2.7}$$

with $\phi$ given by (2.6). If it has solutions other than $c_1 = c_2 = 0$, then $\lambda_\omega$ must satisfy the equality

$$J_{1/3}(\tfrac{2}{3}\lambda^{3/2})J_{-1/3}(\tfrac{2}{3}(\lambda - \overline{L})^{3/2}) - J_{-1/3}(\tfrac{2}{3}\lambda^{3/2})J_{1/3}(\tfrac{2}{3}(\lambda - \overline{L})^{3/2}) = 0. \tag{2.8}$$

So as to avoid annoyances at $\lambda = 0$ and $\lambda = \overline{L}$, we can better rephrase our problem to that of finding the positive zeros of

$$f_{\overline{L}}(\lambda) \equiv \frac{J_{1/3}(\frac{2}{3}\lambda^{3/2})}{J_{-1/3}(\frac{2}{3}\lambda^{3/2})} - \frac{J_{1/3}(\frac{2}{3}(\lambda - \overline{L})^{3/2})}{J_{-1/3}(\frac{2}{3}(\lambda - \overline{L})^{3/2})} \tag{2.9}$$

and, in order to look for the Casimir energy, we will use the $\omega$-eigenmode zeta function, given by

$$\zeta_\omega(s) = \sum_\omega \omega^{-s} = E^{-s/3}\zeta_\lambda\left(\frac{s}{2}\right) \tag{2.10}$$

where the $\lambda$-zeta function is

$$\zeta_\lambda(t) \equiv \sum_\lambda \lambda^{-t} \qquad \mathrm{Re}\, t > 1. \tag{2.11}$$

Hence, the zeta-renormalized Casimir energy [13] is given by

$$E_{\mathrm{C}} = \mathrm{PP}_{s\to-1}\frac{\mu}{2}\zeta_{\omega/\mu}(s) = \mathrm{PP}_{s\to-1}\frac{\mu}{2}E^{-s/3}\zeta_{\lambda/\mu^2}\left(-\frac{s}{2}\right) = \mathrm{PP}_{s\to-1}\frac{\mu}{2}\left(\frac{\mu}{E^{1/3}}\right)^s\zeta_\lambda\left(-\frac{s}{2}\right) \tag{2.12}$$

$\mu$ denoting an arbitrary mass scale. In general, this quantity gives the amount of vacuum energy per real degree of freedom of the field, while the total energy is obtained by multiplying it by an adequate factor (e.g. two for a complex scalar field). Some aspects of the relations between this and other forms of regularization are illustrated in [14–16]. As we see, at $s = -1$, the argument of $\zeta_\lambda$ is $-1/2$. This is where we have to work, i.e. we will have to find the analytic continuation of $\zeta_\lambda(t)$, initially defined by (2.11), to $\mathrm{Re}\, t = -1/2$ (at least for points on the real axis).

We start from the integral representation of $\zeta_\lambda(s)$

$$\zeta_\lambda(s) = \frac{s}{2\pi \mathrm{i}} \int_C \mathrm{d}z\, z^{-s-1} \ln[f_{\overline{L}}(z)] \qquad \mathrm{Re}\, s > 1 \tag{2.13}$$

with the contour $C$ enclosing all the positive real zeros of $f_{\overline{L}}$. Next, we seek an adequate realization of $C$ convenient for numerically calculating the analytic continuation to $s = -1/2$. For non-integer $\nu$, $J_\nu(z)$ has a branch point at $z = 0$ with a cut from this point to infinity in any direction. Looking a bit more closely at what happens around the origin, we see that

$$\frac{J_{1/3}(\frac{2}{3}z^{3/2})}{J_{-1/3}(\frac{2}{3}z^{3/2})} \sim \frac{1}{3^{2/3}} \frac{\Gamma(2/3)}{\Gamma(4/3)} z$$

which is an entire function. Therefore, on the whole there are no cuts to worry about.

Our strategy will be based on the following. Let $A_{\overline{L}}(z)$ denote the asymptotic behaviour for large $|z|$ (under whatever specific conditions) of $f_{\overline{L}}(z)$. We can write

$$\zeta_\lambda(s) = \frac{s}{2\pi \mathrm{i}} \left\{ \int_C \mathrm{d}z\, z^{-s-1} \ln\left[\frac{f_{\overline{L}}(z)}{A_{\overline{L}}(z)}\right] + \int_C \mathrm{d}z\, z^{-s-1} \ln[A_{\overline{L}}(z)] \right\}. \tag{2.14}$$

The advantages of this approach are, first, that since $f_{\overline{L}}(z)/A_{\overline{L}}(z) \to 1$ as $|z| \to \infty$, the first integral will have good properties. Second, if we are able to find an $A_{\overline{L}}(z)$ without zeros or poles inside $C$, the second integral will vanish and we may ignore it; however, if $A_{\overline{L}}(z)$ has zeros in the interior, they will have to be taken into account. We draw our integration circuit so that the interior of $C$ is bounded by two arcs, $|z| = \varepsilon$ and $|z| = R$ (with $\varepsilon \to 0$ and $R \to \infty$), and two radial lines, $\arg z = \pi/3$ and $\arg z = -\pi/3$.

Next we consider our choice of $A_{\overline{L}}(z)$. We get

$$\begin{aligned} f_{\overline{L}}(z) &\sim B_{\overline{L}}(z) \left[1 + O\left(\frac{1}{z^{3/2}}, \frac{1}{(z-\overline{L})^{3/2}}, \ldots\right)\right] \\ B_{\overline{L}}(z) &\equiv \sqrt{3} \frac{\sin[\frac{2}{3}z^{3/2} - \frac{2}{3}(z-\overline{L})^{3/2}]}{\cos[\frac{2}{3}z^{3/2} + \frac{2}{3}(z-\overline{L})^{3/2} + \pi(\frac{1}{3} - \frac{1}{2})] + \cos[\frac{2}{3}z^{3/2} - \frac{2}{3}(z-\overline{L})^{3/2}]}. \end{aligned} \tag{2.15}$$

Hence, one feels tempted to take $A_{\overline{L}}(z) = B_{\overline{L}}(z)$ since, for $|z| \gg 1$,

$$\int \mathrm{d}z\, z^{-s-1} \ln\left[\frac{f_{\overline{L}}(z)}{A_{\overline{L}}(z)}\right] = \int \mathrm{d}z\, z^{-s-1} \ln\left[1 + \mathcal{O}\left(\frac{1}{z^{3/2}}\right)\right] = \mathcal{O}\left(\int \mathrm{d}z\, z^{-s-1} \frac{1}{z^{3/2}}\right) \tag{2.16}$$

which, at $s = -1/2$, is integrable for $z \to \infty$ without problems. However, for $B_{\overline{L}}(z)$ as it stands, the calculation of its zeros on the real axis is complicated. We now try to further simplify the expression by looking at what happens on the straight tracks.

In reality, for real $s$, the straight track with $\arg|z| = -\pi/3$ gives the complex conjugate to that for $\arg|z| = \pi/3$, with opposite sign. The result of their sum will then be 2i times the imaginary part of the latter. In fact we may note that the asymptotic behaviour of $A_{\overline{L}}$ on the part where $\arg|z| = \pi/3$ (say $z = \mathrm{e}^{\mathrm{i}\pi/3}x$†), reduces, up to exponentially small contributions, to the simpler expression

$$\begin{aligned} B_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/3}x) &\sim \sqrt{3}\, \mathrm{e}^{i\pi/3} (\mathrm{e}^{-2\frac{2}{3}(x-\tilde{L})^{3/2}} - \mathrm{e}^{-2\frac{2}{3}x^{3/2}}) \\ &= \sqrt{3}\, \mathrm{e}^{i\pi/3}\, \mathrm{e}^{-\frac{4}{3}x^{3/2}} (\mathrm{e}^{2\tilde{L}x^{1/2}} - 1) \left[1 - \frac{1}{2}\frac{\tilde{L}^2}{x^{1/2}} + O\left(\frac{\tilde{L}^2}{x^{1/2}}\, \mathrm{e}^{-2\tilde{L}x^{1/2}}, \ldots\right)\right] \end{aligned} \tag{2.17}$$

† Note that this $x$ is just a parameter for describing a given line in the complex plane, and has nothing to do with the initial $x$ in our one-dimensional configuration space.

where

$$\tilde{L} \equiv \mathrm{e}^{-\mathrm{i}\pi/3}\overline{L}. \tag{2.18}$$

Then, we adopt

$$A_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/3}x) = \sqrt{3}\,\mathrm{e}^{\mathrm{i}\pi/3}\,\mathrm{e}^{-\frac{4}{3}x^{3/2}}(\mathrm{e}^{2\tilde{L}x^{1/2}} - 1) \tag{2.19}$$

and extend it by considering

$$A_{\overline{L}}(z) = \sqrt{3}\,\mathrm{e}^{\mathrm{i}\pi/3}\,\mathrm{e}^{\mathrm{i}\frac{4}{3}z^{3/2}}(\mathrm{e}^{-\mathrm{i}2\overline{L}z^{1/2}} - 1). \tag{2.20}$$

As a result of (2.15) to (2.19), we get

$$\ln[\mathcal{L}_{\overline{L}}(x)] \equiv \ln\left[\frac{f_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/3}x)}{A_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/3}x)}\right] = \ln\left[1 - \frac{1}{2}\frac{\tilde{L}^2}{x^{1/2}} + \cdots\right] = -\frac{\tilde{L}^2}{2}\frac{1}{x^{1/2}} + \text{higher-order terms} \tag{2.21}$$

where the higher-order terms cause no problem on integrating them. The leading part, however, gives rise to

$$\int^{\infty} \mathrm{d}x\, x^{-s-1}\ln[\mathcal{L}_{\overline{L}}(x)] \sim -\frac{\tilde{L}^2}{2}\int^{\infty} \mathrm{d}x\, x^{-s-1}\frac{1}{x^{1/2}} \tag{2.22}$$

i.e. a logarithmic divergence at $s = -1/2$ when the upper integration bound goes to infinity. We shall parametrize this divergence by performing a subtraction in the integrand and separately adding the subtracted part. In view of (2.22), the subtracted piece will be $-(\tilde{L}^2/2)1/(1+x^2)^{1/4}$, which has the same behaviour as $x \to \infty$ and in addition is integrable around $x = 0$. After integrating,

$$\begin{aligned}
&\int_0^{\infty} \mathrm{d}x\, x^{-s-1}\ln[\mathcal{L}_{\overline{L}}(x)] = \mathcal{J}_{\overline{L}}(s) - \frac{\tilde{L}^2}{4}B\left(-\frac{s}{2}, \frac{s+1/2}{2}\right)\\
&\mathcal{J}_{\overline{L}}(s) \equiv \int_0^{\infty} \mathrm{d}x\, x^{-s-1}\left\{\ln[\mathcal{L}_{\overline{L}}(x)] + \frac{\tilde{L}^2}{2}\frac{1}{(1+x^2)^{1/4}}\right\}
\end{aligned} \tag{2.23}$$

where the main point is that now $\mathcal{J}_{\overline{L}}(s)$ is a finite integral at $s = -1/2$.

The zeros of $A_{\overline{L}}(z)$ on the positive real axis will be the $x$ values satisfying $\mathrm{e}^{-\mathrm{i}2\overline{L}x^{1/2}} - 1 = 0$, i.e. $x = x_n = (\pi n/\overline{L})^2$, $n = 1, 2, 3, \ldots$. As a result, the contribution of the second integral in (2.14), when $C$ encloses the *whole* positive real axis, is

$$\frac{s}{2\pi\mathrm{i}}\int_C \mathrm{d}z\, z^{-s-1}\ln[A_{\overline{L}}(z)] = \sum_{n=1}^{\infty} x_n^{-s} = \left(\frac{\pi}{\overline{L}}\right)^{-2s}\zeta_R(2s). \tag{2.24}$$

Concerning the remaining parts of the circuit $C$, for the arc of radius $\varepsilon$ the result of integrating $z^{-s-1}\ln[f_{\overline{L}}(z)/A_{\overline{L}}(z)]$ over this part, when $\varepsilon \to 0$, amounts to bounded quantities times $\varepsilon^{-s}$ or $\varepsilon^{-s}\ln\varepsilon$. Therefore, for $s = -1/2$ one can ignore this piece of the circuit. As for the arc of radius $R \to \infty$, by (2.16), we may also throw away this part when $s = -1/2$.

Thus, for $x \in (0, \infty)$, having already taken the limits $\varepsilon \to 0$ and $R \to \infty$, we obtain

$$\begin{aligned}
\zeta_\lambda(s) &= -\frac{s}{2\pi\mathrm{i}}2\mathrm{i}\,\mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\pi s/3}\int_0^{\infty}\mathrm{d}x\, x^{-s-1}\ln[\mathcal{L}_{\overline{L}}(x)]\right\} + \left(\frac{\pi}{\overline{L}}\right)^{-2s}\zeta_R(2s)\\
&= -\frac{s}{\pi}\mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\pi s/3}\left[\mathcal{J}_{\overline{L}}(s) - \frac{\tilde{L}^2}{4}B\left(-\frac{s}{2}, \frac{s+1/2}{2}\right)\right]\right\} + \left(\frac{\pi}{\overline{L}}\right)^{-2s}\zeta_R(2s)\\
&= -\frac{s}{\pi}\left[\overline{\mathcal{J}_{\overline{L}}}(s) + \frac{\overline{L}^2}{4}\sin\left(\frac{\pi}{3}(s+2)\right)B\left(-\frac{s}{2}, \frac{s+1/2}{2}\right)\right] + \left(\frac{\pi}{\overline{L}}\right)^{-2s}\zeta_R(2s)
\end{aligned} \tag{2.25}$$

where the term

$$\overline{\mathcal{J}_{\overline{L}}}(s) \equiv \mathrm{Im}\,[\mathrm{e}^{-\mathrm{i}\pi s/3}\mathcal{J}_{\overline{L}}(s)]$$
$$= \int_0^\infty \mathrm{d}x\, x^{-s-1}\left\{\mathrm{Im}[\mathrm{e}^{-\mathrm{i}\pi s/3}\mathcal{L}_{\overline{L}}(x)] - \frac{\overline{L}^2}{2}\sin\left(\frac{\pi}{3}(s+2)\right)\frac{1}{(1+x^2)^{1/4}}\right\} \tag{2.26}$$

is also a finite integral at $s=-1/2$. Laurent expanding (2.25) around $s=-1/2$,

$$\zeta_\lambda(s) = \frac{\overline{L}^2}{4\pi}\frac{1}{s+\frac{1}{2}} + \frac{1}{2\pi}\left[-\frac{\overline{L}^2}{4}\left(\gamma+\psi\left(-\frac{1}{4}\right)\right) + \overline{\mathcal{J}_{\overline{L}}}\left(-\frac{1}{2}\right)\right] - \frac{\pi}{12\overline{L}} + \mathcal{O}\left(s+\frac{1}{2}\right) \tag{2.27}$$

and, after applying (2.12), it is immediate that

$$\frac{E_{\mathrm{C}}}{E^{1/3}}(\mu;\overline{L}) = -\frac{\overline{L}^2}{8\pi}\ln\left(\frac{\mu}{E^{1/3}}\right) + p_{\overline{L}}$$
$$p_{\overline{L}} \equiv -\frac{\pi}{24}\frac{1}{\overline{L}} + \frac{1}{4\pi}\left[-\frac{\overline{L}^2}{4}\left(\gamma+\psi\left(-\frac{1}{4}\right)\right) + \overline{\mathcal{J}_{\overline{L}}}\left(-\frac{1}{2}\right)\right]. \tag{2.28}$$

One can easily see that around $\overline{L}=0$ the main contribution comes from the term $-\pi/24\overline{L}$. In fact, when $E=0$ ($\overline{L}=0$) the energy becomes $-\pi/24L$, which is $s$-finite and coincides with the known value in absence of external field (see, e.g., [2, 3]†).

By numerical evaluation of the integral (2.26) at $s=-1/2$, and setting the arbitrary mass scale at $\mu=E^{1/3}$, we find the finite parts of $LE_{\mathrm{C}}(\mu=E^{1/3};\overline{L}) = \overline{L}p_{\overline{L}}$ for different values of $\overline{L}$. The results of our calculation are listed in table 1. Note the presence of a local minimum around $\overline{L}=0.3$. As $\overline{L}$ is further raised, this $LE_{\mathrm{C}}$ increases and eventually becomes positive. Yet attributing physical meaning to such facts would be difficult, as these values are scale dependent and not completely unambiguous. In order to grasp their importance in numerical terms, let us imagine that $V(x)=\alpha mgx$ with $m$, $g$ and $x$ denoting mass, gravity acceleration on the Earth surface and height, respectively, in m.k.s. units. Then, for dimensional reasons, by looking at the original equation written in this unit system we realize that the coupling $\alpha$ must have the same dimensions as $M_{\mathrm{p}}/\hbar^2$, with $M_{\mathrm{p}}$ the Planck mass. This coupling then becomes completely determined up to a dimensionless constant which we make equal to one unit; then $\alpha=M_{\mathrm{p}}/\hbar^2$, and we find that for $m=10^{-30}$ kg (close to the electron mass) the value $\overline{L}=0.5$ means $L=1.9\times10^{-11}$ m. In such circumstances, the change in $LE_{\mathrm{C}}$ with respect to the case without external field amounts to 3%, approximately.

**Table 1.** Numerical values of $LE_{\mathrm{C}}$ in terms of the dimensionless variable $\overline{L}$.

| $\overline{L}$ | $LE_{\mathrm{C}}(\mu=E^{1/3};\overline{L})$ |
|---|---|
| 0 | $-\pi/24 \simeq -0.131$ |
| 0.1 | $-0.134$ |
| 0.2 | $-0.136$ |
| 0.3 | $-0.137$ |
| 0.4 | $-0.136$ |
| 0.5 | $-0.135$ |

† In [2] the expression corresponding to this result has to be corrected.

### *2.2. Case of Dirichlet boundary conditions at x = 0 and x → ∞*

Here we shall consider the limit $\overline{L} \to \infty$ while keeping $\overline{L} > \lambda$. We thus write $\overline{L} - \lambda \equiv u > 0$. The boundary condition, at $x \to \infty$ ($y \to \infty$ with positive values) lead one to consider the identity

$$J_\nu\left(\tfrac{2}{3}(-u)^{3/2}\right) = \mathrm{e}^{\mathrm{i}3\pi\nu/2} I_\nu\left(\tfrac{2}{3}u^{3/2}\right) \tag{2.29}$$

and then

$$c_1 J_{1/3}\left(\frac{2}{3}(-u)^{3/2}\right) + c_2 J_{-1/3}\left(\frac{2}{3}(-u)^{3/2}\right)$$
$$= c_1\,\mathrm{e}^{\mathrm{i}\pi/2} I_{1/3}\left(\frac{2}{3}u^{3/2}\right) + c_2\,\mathrm{e}^{-\mathrm{i}\pi/2} I_{-1/3}\left(\frac{2}{3}u^{3/2}\right) \sim \mathrm{i}(c_1 - c_2)\frac{1}{\sqrt{2\pi\,\frac{2}{3}u^{3/2}}}\,\mathrm{e}^{\frac{2}{3}u^{3/2}}. \tag{2.30}$$

Since this must vanish, the relation

$$c_2 = c_1 \tag{2.31}$$

is now needed, and the vanishing condition of $\phi$ at $x = 0$ ($y = -\lambda$) becomes, by (2.6),

$$J_{1/3}\left(\tfrac{2}{3}\lambda^{3/2}\right) + J_{-1/3}\left(\tfrac{2}{3}\lambda^{3/2}\right) = 0. \tag{2.32}$$

Equivalently, the eigenmodes correspond to the zeros of

$$f_\infty(\lambda) = \frac{J_{1/3}\left(\frac{2}{3}\lambda^{3/2}\right)}{J_{-1/3}\left(\frac{2}{3}\lambda^{3/2}\right)} + 1. \tag{2.33}$$

On the path $\lambda = \mathrm{e}^{\mathrm{i}\pi/3}x$,

$$f_\infty(\mathrm{e}^{\mathrm{i}\pi/3}x) = \mathrm{e}^{\mathrm{i}\pi/3}\frac{I_{1/3}\left(\frac{2}{3}x^{3/2}\right)}{I_{-1/3}\left(\frac{2}{3}x^{3/2}\right)} + 1 \tag{2.34}$$

which, for $x \to \infty$, has a dominant contribution given by a constant, namely

$$A_\infty = \mathrm{e}^{\mathrm{i}\pi/3} + 1. \tag{2.35}$$

The energy here will be the result of applying (2.14) with the same $C$, and $f_\infty$ and $A_\infty$ instead of $f_{\overline{L}}$ and $A_{\overline{L}}$. Note that, since $A_\infty$ has no zeros, the second integral now gives no contribution. Next-to-leading terms cause no problem in the sense that, unlike expression (2.21), the asymptotic expansion of

$$\ln[\mathcal{L}_\infty(x)] \equiv \ln\left[\frac{f_\infty(\mathrm{e}^{\mathrm{i}\pi/3}x)}{A_\infty}\right]$$

contains no term $\sim x^{-1/2}$, and it is enough to numerically calculate (for real $s$)

$$\zeta_\lambda(s) = -\frac{s}{\pi}\,\mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\pi s/3}\int_0^\infty \mathrm{d}x\, x^{-s-1}\ln[\mathcal{L}_\infty(x)]\right\}. \tag{2.36}$$

After numerical evaluation of the integral at $s = -1/2$, we use (2.12) and find

$$E_{\mathrm{C}} = -0.088\,346 E^{1/3}. \tag{2.37}$$

The above finite figure comes as a surprise in view of the divergent nature of the finite-$\overline{L}$ results. In other words, this value of $E_{\mathrm{C}}$ cannot be obtained as the limit of the finite-$\overline{L}$ outcome when $\overline{L}$ goes to infinity. This is so because the imposition of the boundary condition at infinity in the above-described way is actually a qualitatively different situation. In the finite-$\overline{L}$ case there is room for 'external' modes propagating in the region $L \leqslant x < \infty$ which were not included because we were assuming our field modes 'confined' in the $0 \leqslant x \leqslant L$ region. In contrast, the infinite-$\overline{L}$ case leaves no room for these external modes. Considering analogous discussions for the Casimir effect in higher (odd) dimensions, it is conceivable that the inclusion of external modes might lead to the cancellation of the $s$-singularities and the appearance of (2.37) as a continuous limit of finite values. The proof of this conjecture is out of the scope of the present work.

Next, let us compare the figure found with the result of an approximate calculation. By a 'WKB-style' approximation, we write

$$J_\nu(x) + J_{-\nu}(x) \sim 2\cos\left(\frac{\pi\nu}{2}\right)\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\pi}{4}\right) \tag{2.38}$$

whose zeros are $x_n = (n + \frac{3}{4})\pi$, $n = 0, 1, 2, \ldots$. Therefore, in terms of $\lambda$

$$\lambda_n^{\mathrm{WKB}} = [\tfrac{3}{2}(n + \tfrac{3}{4})\pi]^{2/3} \tag{2.39}$$

and

$$\zeta_\lambda^{\mathrm{WKB}}(s) = (\tfrac{3}{2}\pi)^{-2s/3}\zeta_H(s, \tfrac{3}{4}). \tag{2.40}$$

Evaluating the energy at $s = -1/2$ we find

$$E_{\mathrm{C}}^{\mathrm{WKB}} = \tfrac{1}{2}\zeta_\lambda^{\mathrm{WKB}}(-\tfrac{1}{2})E^{1/3} = \tfrac{1}{2}(\tfrac{3}{2}\pi E)^{1/3}\zeta_H(-\tfrac{1}{3}, \tfrac{3}{4}) = -0.075\,0921 E^{1/3} \tag{2.41}$$

which is not too different from (2.37). The fact that a WKB-fashion approximation is fairly good for $D = 1$ seems to be in accordance with the ideas of [7], which shows a case where the approximation worsens as $D$ increases.

## 3. Charged scalar in an external electric field

A charged scalar in an external electric field was the situation considered in [1] and, by way of comparison, we shall sketch here the changes that take place. The evolution equation (for the massless case) is

$$D^2\Phi(t, x) = 0. \tag{3.1}$$

Here we have covariant derivatives $D_\mu = \partial_\mu + \mathrm{i}qA_\mu$, where $q$ is the charge and $A$ the electromagnetic potential for the external field. In view of the type of field supposed, we take $A_1 = 0$ and $A_0 = Ex + \text{constant}$. $E$ denotes the value of our uniform field. The Dirichlet boundary condition will eventually be imposed at $x = 0$ and $x = L$. To make our expressions similar to those in the above quoted reference, we choose the constant $= -EL/2$. As usual, we write $\Phi(t, x) = \mathrm{e}^{-\mathrm{i}\omega t}\phi(x)$. Then, in terms of the new dimensionless variables $u \equiv x/L$, $\overline{L} \equiv qEL^2$ and $w \equiv L\omega$, the field equation reads

$$\left[\frac{\mathrm{d}^2}{\mathrm{d}u^2} + \left(\overline{L}\left(u - \frac{1}{2}\right) + w\right)^2\right]\phi = 0. \tag{3.2}$$

Its general solution is a linear combination

$$\phi = y^{1/2}\left[c_1 J_{1/4}\left(\frac{y^2}{2\overline{L}}\right) + c_2 J_{-1/4}\left(\frac{y^2}{2\overline{L}}\right)\right] \qquad y \equiv \overline{L}\left(u - \frac{1}{2}\right) + w. \tag{3.3}$$

When imposing the Dirichlet boundary condition at $x = 0, a$ $(u = 0, 1)$, one realizes that the eigenfrequencies will be given by the solutions of

$$f_{\overline{L}}(w) \equiv \frac{J_{1/4}((1/2\overline{L})(w - (\overline{L}/2))^2)}{J_{-1/4}((1/2\overline{L})(w - (\overline{L}/2))^2)} - \frac{J_{1/4}((1/2\overline{L})(w + (\overline{L}/2))^2)}{J_{-1/4}((1/2\overline{L})(w + (\overline{L}/2))^2)} = 0. \tag{3.4}$$

As an integration contour, we take the same $C$ as in the previous problem but with a semi-angle of $\pi/4$ instead of $\pi/3$, which proves to be more convenient for the present type of integrand. The study of $f_{\overline{L}}$ on the upper straight track leads to the consideration of an asymptotic form for $f_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/4}x)$ for $x$ large. The one we adopt is

$$A_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/4}x) \equiv \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}\left\{\exp\left[-\frac{1}{\overline{L}}\left(x+\frac{\mathrm{e}^{-\mathrm{i}\pi/4}\overline{L}}{2}\right)^2\right]-\exp\left[-\frac{1}{\overline{L}}\left(x-\frac{\mathrm{e}^{-\mathrm{i}\pi/4}\overline{L}}{2}\right)\right]^2\right\} \tag{3.5}$$

which is regarded as a particular case of the complex function

$$A_{\overline{L}}(z) \equiv \sqrt{2}\,\mathrm{e}^{\mathrm{i}\pi/4}[\mathrm{e}^{\mathrm{i}(z+\overline{L}/2)^2/\overline{L}} - \mathrm{e}^{\mathrm{i}(z-\overline{L}/2)^2/\overline{L}}] \tag{3.6}$$

asymptotically mimicking $f_{\overline{L}}(z)$ for large $|z|$. This function has real zeros—contributing to the second integral in (2.14)—which are $x_n = \pi n$, $n = 1, 2, 3, \ldots$. Taking all these elements into account, we find

$$\zeta_w(s) = -\frac{s}{\pi}\overline{\mathcal{J}_{\overline{L}}}(s) + \pi^{-s}\zeta_R(s) \tag{3.7}$$

where

$$\overline{\mathcal{J}_{\overline{L}}}(s) = \mathrm{Im}\left\{\mathrm{e}^{-\mathrm{i}\pi s/4}\int_0^\infty \mathrm{d}x\, x^{-s-1}\ln[\mathcal{L}_{\overline{L}}(x)]\right\} \qquad \ln[\mathcal{L}_{\overline{L}}(x)] \equiv \ln\left[\frac{f_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/4}x)}{A_{\overline{L}}(\mathrm{e}^{\mathrm{i}\pi/4}x)}\right]. \tag{3.8}$$

Setting $s = -1$ and adequately re-introducing the quantities with dimensions, we obtain the Casimir energy

$$E_{\mathrm{C}} = 2\frac{1}{2L}\zeta_w(-1) = \frac{1}{L}\left[-\frac{\pi}{12}+\frac{1}{\pi}\overline{\mathcal{J}_{\overline{L}}}(-1)\right]. \tag{3.9}$$

Since the space-dependent part of the field ($\phi(x)$) is now necessarily complex, an extra factor of two has been introduced. While the first term is the known result in the absence of external field, the integral $\overline{\mathcal{J}_{\overline{L}}}(-1)$, which is finite and gives the new contribution due to $\overline{L}$, has to be numerically evaluated for every $\overline{L}$. The results are given in table 2. The m.k.s. form of the initial equation leads to the relation $\overline{L} = (1/\hbar c)qEL^2$. Then, taking $q$ as the electron charge and $L = 10^{-7}$ m, we see that at $\overline{L} = 0.5$ the field intensity $E$ is close to $10^7(N/C)$. In such conditions, $LE_{\mathrm{C}}$ is 46% larger than in the absence of electric field. This is obviously a very appreciable effect from the experimental point of view.

**Table 2.** Numerical values of $LE_{\mathrm{C}}$ as a function of the dimensionless variable $\overline{L}$.

| $\overline{L}$ | $LE_{\mathrm{C}}(\overline{L})$ |
|---|---|
| 0 | $-\pi/12 \simeq -0.262$ |
| 0.1 | $-0.312$ |
| 0.2 | $-0.334$ |
| 0.3 | $-0.352$ |
| 0.4 | $-0.368$ |
| 0.5 | $-0.382$ |

Soft-boundary ‘potentials’, quadratic in the position $V(\boldsymbol{x}) \propto \sum_i \alpha_i^4 x_i^2$, were studied in [12], where other ways of ‘hardening’ the endpoints were also considered. Up to a shift in the origin, the wave equations with the $V(\boldsymbol{x})$-terms have essentially the same form as equation (3.2) above. Our figures are in agreement with the facts, noted in [12], that the

Casimir effect is attractive (minus sign in the above results) and that the energy is further decreased as the value of the 'coupling' ($\bar{L}$, or $\alpha_i$) is increased.

The case where the Dirichlet boundary conditions are set at $x = 0$ and $x \to \infty$ may be dealt with by a method analogous to the one employed in the previous problem. The final result is

$$E_{\mathrm{C}} = \sqrt{qE}\,\frac{1}{\pi}\,\mathrm{Im}\left\{\mathrm{e}^{\mathrm{i}\pi/4}\int_0^\infty \mathrm{d}x\,\ln\left[\frac{\mathrm{e}^{\mathrm{i}\pi/4}(I_{1/4}(x^2/2)/I_{-1/4}(x^2/2)) + 1}{\mathrm{e}^{\mathrm{i}\pi/4}+1}\right]\right\}$$
$$= -0.160\,314\sqrt{qE}. \qquad (3.10)$$

## 4. Conclusions

Some precise ideas follow from this study. The general conclusion that apparently simple situations can become so mathematically involved (compelling us to be very meticulous at every slippery step) has found its compensation in the reassuring feeling that the zeta-function method, conveniently supplied with complex analytical techniques, is a very powerful tool to deal with such problems.

The second idea comes from the numerical results themselves. Forgetting for a second that the spacetime models considered here are two-dimensional, we would conclude that an experimental verification of these theories stays within the reach of present or near-future settings. And this, we feel, is quite a respectable goal, since a clear understanding of the concept of zero-point energy is of fundamental importance and a necessary clue for the adequate comprehension of the quantum field theories themselves.

There are two tasks to be approached next. On the one hand, supported by the present two-dimensional results, the physical situation of four spacetime dimensions ought to be investigated. On the other hand, the standpoint of the semiclassical model adopted here for the gravitational case (an example, in fact, of the general class of semihard boundary conditions) should be substantially improved in order to approach a more realistic (and computationally demanding) quantum gravity approximation.

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